Tetration Truncation - Solution

Note that determining the last k digits of $a \uparrow\uparrow n$ is equivalent to finding $a \uparrow\uparrow n \pmod{10^k}$,potentially padding the value with additional zeroes to fill all k digits. We use a recursive approach to make use of the recursive structure of tetration. In particular, for $n \in \mathbb{N}$ we have

$$a \uparrow\uparrow n = a^{a \uparrow\uparrow (n-1)}.$$

From Euler's Totient Theorem we know that for $\ell \in \mathbb{N}$ coprime with a, we have

$$a^{\varphi(\ell)} \equiv 1 \pmod{\ell}.$$

In particular, if we know that $a \uparrow\uparrow (n-1) \equiv r \pmod{\varphi(\ell)}$, we can write that $a \uparrow\uparrow (n-1) = q\varphi(\ell) + r$ and then it follows that

$$a \uparrow\uparrow n = a^{a \uparrow\uparrow (n-1)} = a^{q\varphi(\ell)+r} = \left(a^{\varphi(\ell)}\right)^q a^r \equiv a^r \pmod{\ell}.$$

Hence, our approach will be to recursively compute

$$a \uparrow\uparrow (n-s) \pmod{\underbrace{\varphi(\ldots\varphi}_{s \text{ times}}(10^k)\ldots)}.$$

The base case of $a \uparrow\uparrow 1 = a$ is handled simply by computing the residue, which is feasible as a is not too large. In any other case we reduce the tetration by one level by the relation above. The only additional thing to keep in mind is that we need to update the modulus by applying another φ. To factor the modulus each time would take too long. Instead, we keep track of all the prime factors of the modulus. Initially, we can easily factor the modulus as $10^k = 2^k 5^k$. Now applying φ yields

$$\varphi(10^k) = 2^{k-1} \cdot 4 \cdot 5^{k-1} = 2^{k+1} 5^{k-1},$$

and as long as both exponents stay positive, we can apply this same rule. If one of the exponents is zero, then we need a slightly different result, but this is also easily derived. In this way, we can easily keep track of the respective modulus throughout the recursion. A key observation is that the modulus will always only have factors 2 and 5, which means that all our previous applications of Euler's Totient Theorem were justified as $\gcd(a, 10) = 1$, and so a, and any tetration of a, has no factors 2 and 5.

See the solution code for implementation specifics.